GEOMETRIC PROPERTIES OF THE FAMILY OF p-PARALLEL BODIES

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Abstract. We study geometric properties of the family of p-parallel bodies of a convex body K with respect to a gauge body E. In particular, we investigate various regularity properties of their boundaries by means of their 0-extreme vectors, aiming for extensions of several results known for the $p = 1$ case. We also analyze decomposition properties of convex bodies via the p-sum of their p-inner parallel bodies. To this end, we introduce a new convex body associated with K, which is related to the p-parallel bodies and the p-sum of convex bodies, the p-form body, and examine its properties. The latter provides improvements of inequalities involving p-inner parallel bodies, their support functions and mixed volumes.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., non-empty compact convex sets in the Euclidean space, endowed with the standard scalar product $\langle \cdot, \cdot \rangle$, and the Euclidean metric. Let $\mathcal{K}^n_0$ be the subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin. We also denote by $\mathcal{K}^n_n$ (respectively, $\mathcal{K}^n_{(0)}$) the subset of $\mathcal{K}^n$ having interior points (0 as an interior point). A convex body K is called *strictly convex* if its boundary $\operatorname{bd} K$ does not contain a segment, and *regular* if all its boundary points are regular, i.e., the supporting hyperplane to K at any $x \in \operatorname{bd} K$ is unique. Let B_n be the n-dimensional unit ball and $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$. The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\operatorname{vol}(M)$, its interior by $\operatorname{int} M$, its closure by $\operatorname{cl} M$, and its convex and linear hull by $\operatorname{conv} M$ and $\operatorname{lin} M$, respectively. The dimension of M, i.e., the dimension of its affine hull, is denoted by $\dim M$. If μ is a Borel measure on $\mathbb{S}^{n-1}$, its support is denoted by $\operatorname{supp} \mu$.

The Minkowski addition and its counterpart the Minkowski difference of non-empty sets in $\mathbb{R}^n$ are defined as

$$A + B = \{a + b : a \in A, b \in B\}, \text{ and } A \sim B = \{x \in \mathbb{R}^n : B + x \subseteq A\},$$

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respectively. We refer the reader to the book [32, Section 3.1] for a detailed study of them.

In 1962 Firey introduced the following generalization of the classical Minkowski addition (see [9]). For $1 \leq p < \infty$ and $K, E \in \mathcal{K}_0^n$, the p-sum (or L_p sum) of K and E is the convex body $K +_p E \in \mathcal{K}_0^n$ defined as follows:

$$h(K +_p E, u) = \Bigl(h(K, u)^p + h(E, u)^p\Bigr)^{1/p},$$

for all $u \in \mathbb{S}^{n-1}$, where $h(K, u) = \max\bigl\{\langle x, u\rangle : x \in K\bigr\}$ denotes the support function of $K \in \mathcal{K}^n$ in the direction $u \in \mathbb{S}^{n-1}$ (see [32, Section 1.7] for a detailed study of the support function).

In [28] the following counterpart of the p-sum was introduced: for $K, E \in \mathcal{K}_0^n$, $E \subseteq K$, and $1 \leq p < \infty$, the *p-difference* of K and E is defined as

$$K \sim_p E = \Bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \leq \bigl(h(K, u)^p - h(E, u)^p\bigr)^{1/p},\ u \in \mathbb{S}^{n-1}\Bigr\}.$$

When $p = 1$, in both above cases the usual Minkowski sum and difference are obtained. From the definition it follows that for any $1 \leq p < \infty$

$$h(K \sim_p E, u) \leq \bigl(h(K, u)^p - h(E, u)^p\bigr)^{1/p}. \tag{1.1}$$

When dealing with the p-difference it is useful to work with the following subfamily (see [28] for further details):

$$\mathcal{K}_{00}^n(E) = \bigl\{K \in \mathcal{K}_0^n : 0 \in K \sim \mathrm{r}(K; E)E\bigr\},$$

where

$$\mathrm{r}(K; E) = \max\bigl\{r \geq 0 : x + rE \subseteq K \text{ for some } x \in \mathbb{R}^n\bigr\}$$

is the relative inradius of K with respect to E.

Let $E \in \mathcal{K}_0^n$ and $K \in \mathcal{K}_{00}^n(E)$. The *full system of p-parallel bodies* of K relative to E is defined as follows.

Definition 1.1 ([28]). Let $E \in \mathcal{K}_0^n$, and let $K \in \mathcal{K}_{00}^n(E)$. For $1 \leq p < \infty$,

$$K_\lambda^p = \begin{cases} K \sim_p |\lambda| E & \text{if } \ -\mathrm{r}(K; E) \leq \lambda \leq 0, \\ K +_p \lambda E & \text{if } \qquad\qquad 0 \leq \lambda < \infty. \end{cases}$$

K_λ^p is the *p-inner* (respectively, *p-outer*) parallel body of K at distance $|\lambda|$ relative to E and $\ker_p(K; E) := K_{-\mathrm{r}(K;E)}^p$ is the *p-kernel of K with respect to E*.

We point out that here the standard notion of inradius is used because, indeed, what would be the natural definition of p-inradius actually coincides with the classical one (see [28, Section 3]).

It is known (see [2, p. 59] for $p = 1$ and [28, Proposition 3.1] for $p > 1$) that the p-kernel of $K \in \mathcal{K}_{00}^n(E)$ with respect to $E \in \mathcal{K}_0^n$ is always a lower-dimensional convex body, for $1 \leq p < \infty$.

For the sake of simplicity, we will omit the index 1 in the above concepts in the classical setting of $p = 1$, and thus, we will write K_λ instead of K_λ^1 or, for instance, will just say *inner* rather than 1-inner.

The following operation introduced in [28] between real numbers plays a relevant role when dealing with the p-sum of convex bodies. Notice that, as also negative reals are considered, this definition extends (up to a constant) the classical p-mean of positive real numbers (see [14]). Let $+_p : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ denote the binary operation defined by

$$(1.2)\quad a +_p b = \begin{cases} \mathrm{sgn}_2(a,b)\left(|a|^p + |b|^p\right)^{1/p} & \text{if } ab \geq 0,\\ \mathrm{sgn}_2(a,b)\Big(\max\big\{|a|,|b|\big\}^p - \min\big\{|a|,|b|\big\}^p\Big)^{1/p} & \text{if } ab \leq 0, \end{cases}$$

being $\mathrm{sgn}_2 : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ the function given by

$$\mathrm{sgn}_2(a,b) = \begin{cases} \mathrm{sgn}(a) = \mathrm{sgn}(b) & \text{if } ab \geq 0,\\ \mathrm{sgn}(a) & \text{if } ab \leq 0 \text{ and } |a| \geq |b|,\\ \mathrm{sgn}(b) & \text{if } ab \leq 0 \text{ and } |a| < |b|, \end{cases}$$

where, as usual, sgn denotes the sign function. We notice that for $ab \geq 0$, this definition corresponds, up to maybe a signed constant, to the classical p-mean ([14, Chapter II]) and does not correspond to any of the more general ϕ-means considered in [14, Chapter III]. We include here the following result from [28], which provides us, among others, with the value of the inradius of the p-inner parallel bodies.

Proposition 1.2 ([28])**.** *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$, and let $\lambda, \mu \geq 0$. The following relations hold for any $1 \leq p < \infty$:*

i) $\left(K^p_\lambda\right)^p_\mu = K^p_{\lambda +_p \mu}$.

ii) $\left(K^p_{-\lambda}\right)^p_\mu \subseteq K^p_{(-\lambda)+_p\mu}$ *if* $\lambda \leq \mathrm{r}(K;E)$.

iii) $\left(K^p_{-\lambda}\right)^p_{-\mu} = K^p_{(-\lambda)+_p(-\mu)}$ *if* $\lambda^p + \mu^p \leq \mathrm{r}(K;E)^p$.

iv) $\left(K^p_\lambda\right)^p_{-\mu} = K^p_{\lambda+_p(-\mu)}$ *if* $\mu \leq \mathrm{r}(K;E) +_p \lambda$.

v) $\lambda K^p_\sigma = \left(\lambda K\right)^p_{\lambda\sigma}$ *for all* $-\mathrm{r}(K;E) \leq \sigma < \infty$.

Notice that iv) yields that

$$\mathrm{r}(K^p_\lambda;E) = \lambda +_p \mathrm{r}(K;E).$$

The following observation will be implicitly used several times. We include it explicitly for completeness.

Remark 1.3. *Let $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$ and* $\mathrm{r} := \mathrm{r}(K;E)$. *We observe that*

$$(1.3)\qquad h(K^p_\lambda,u) \leq \Big(h(K,u)^p - |\lambda|^p h(E,u)^p\Big)^{1/p}$$

for all $-\mathrm{r} \leq \lambda \leq 0$.

For convex bodies $K_1,\dots,K_m \in \mathcal{K}^n$ and real numbers $\lambda_1,\dots,\lambda_m \geq 0$, the volume of the linear combination $\lambda_1K_1 + \dots + \lambda_mK_m$ can be expressed as a homogeneous polynomial in the variables $\lambda_1,\dots,\lambda_m$:

$$(1.4)\qquad \mathrm{vol}(\lambda_1K_1 + \dots + \lambda_mK_m) = \sum_{i_1,\dots,i_n=1}^{m} \mathrm{V}(K_{i_1},\dots,K_{i_n})\lambda_{i_1}\dots\lambda_{i_n},$$

whose coefficients $\mathrm{V}(K_{i_1},\dots,K_{i_n})$ are the *mixed volumes* of $K_1,\dots,K_m$. Moreover, it is known that there exist finite Borel measures on $\mathbb{S}^{n-1}$, the so-called *mixed area measures* $\mathrm{S}(K_2,\dots,K_n,\cdot)$, satisfying that

$$\mathrm{V}(K_1,\dots,K_n)=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h(K_1,u)\,\mathrm{dS}(K_2,\dots,K_n,u).$$

If only two convex bodies $K,E\in\mathcal{K}^n$ are involved in the previous sum, the mixed volumes arising in (1.4), $\mathrm{V}(K[n-i],E[i])=:\mathrm{W}_i(K;E)$, where $K[j]$ denotes that the body K appears j times, are called *quermassintegrals* of K (relative to E). In particular, $\mathrm{W}_0(K;E)=\mathrm{vol}(K)$, $\mathrm{W}_n(K;E)=\mathrm{vol}(E)$. In the case of the mixed area measures, we will also abbreviate the notation writing $\mathrm{S}\bigl(K[n-i-1],E[i],\cdot\bigr)$ if the convex bodies are repeated. For a thorough study on quermassintegrals, mixed volumes and mixed area measures we refer to [32, Chapter 5].

One of the most useful tools to deal with p-inner parallel bodies is the notion of Wulff-shape. If $\Omega\subseteq\mathbb{S}^{n-1}$ is a subset of the unit sphere, not lying in a closed hemisphere, and $\psi:\mathbb{S}^{n-1}\to[0,\infty)$ is a non-negative function, the closed convex set

$$\mathrm{WS}(\Omega,\psi):=\bigcap_{u\in\Omega}\{x\in\mathbb{R}^n:\langle x,u\rangle\le\psi(u)\}$$

is called the *Wulff-shape* or *Aleksandrov body* associated with (Ω,ψ) (or just with ψ, if $\Omega=\mathbb{S}^{n-1}$). We write also $\mathrm{WS}(\psi):=\mathrm{WS}(\mathbb{S}^{n-1},\psi)$. For further details about Wulff-shapes we refer the reader to [32, Section 7.5] and the references therein. For instance, the p-inner parallel bodies of a convex body K (relative to E) can be seen as the Wulff-shape

$$K_\lambda^p=\mathrm{WS}\Bigl(\bigl(h(K,\cdot)^p-|\lambda|^p h(E,\cdot)^p\bigr)^{1/p}\Bigr)$$

(see [32, p. 411] for the case $p=1$).

Remark 1.4. *Using* [32, Lemma 7.5.1 and (7.100)] *we obtain that for any such ψ, $\mathrm{WS}(\psi)$ is a convex body containing the origin, and moreover,*

$$h\bigl(\mathrm{WS}(\psi),\cdot\bigr)\le\psi(\cdot).$$

Furthermore, it is not difficult to prove that $h\bigl(\mathrm{WS}(\psi),u\bigr)=\psi(u)$ for all $u\in\operatorname{supp}\mathrm{S}\bigl(\mathrm{WS}(\psi)[n-1],\cdot\bigr)$.

We recall that a vector $u\in\mathbb{S}^{n-1}$ is a 0-*extreme normal vector* of a convex body $K\in\mathcal{K}^n$ if it cannot be written as $u=u_1+u_2$, with u_i linearly independent normal vectors at one and the same boundary point of K. We write $\mathcal{U}_0(K)$ to denote the set of 0-extreme normal vectors of K. A somehow dual concept arises with the notion of extreme point: $y\in K$ is an *extreme point* of K if it cannot be written in the form $y=(1-\lambda)x+\lambda z$ with $x,z\in K$ and $\lambda\in(0,1)$. Then, the so-called dual of the Krein-Milman theorem (see e.g. [32, Notes for Section 1.4] or [30, (2.9)]) allows us to write

$$K = \bigcap_{u \in \mathcal{U}_0(K)} \big\{ x \in \mathbb{R}^n \,:\, \langle x, u \rangle \leq h(K, u) \big\}. \tag{1.5}$$

The latter establishes, roughly speaking, that in the same manner as extreme points of K determine K by means of the convex hull, extreme vectors of K determine K by means of intersections of halfspaces. These vectors allow one to define, by means of a Wulff-shape, a special convex body associated with two convex bodies: for $K, E \in \mathcal{K}^n_n$, the (relative) *form body* of K *with respect to* E is defined as (see e.g. [6])

$$K^* := \mathrm{WS}\big(\mathcal{U}_0(K), h(E, \cdot)\big).$$

For an alternative (equivalent) definition of form body see [32, p. 386]. Although K^* depends also on the so-called gauge body E, for the sake of simplicity, we omit E in the notation.

Differentiability properties of functions that depend on one-parameter families of convex bodies play an important role in some proofs in Convex Geometry (see e.g. [32, Theorem 7.6.19 and Notes to Section 7.6]). In particular, for $E \in \mathcal{K}^n_n$ and $K \in \mathcal{K}^n$, the differentiability of functions depending on the full-system of parallel bodies was already addressed by Bol (see [1]) and Hadwiger (see [13]). One of the most useful classical tools in this context is the differentiability of the function $\mathrm{vol}(K_\lambda)$ on $-\mathrm{r}(K; E) \leq \lambda \leq 0$, whose derivative is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{vol}(K_\lambda) = n \mathrm{W}_1(K_\lambda; E).$$

The integral representation

$$\mathrm{W}_i(K; E) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u)\, \mathrm{dS}(K[n-i-1], E[i], u) \tag{1.6}$$

leads to think that the differentiability of quermassintegrals is related, in a certain way, with the differentiability of the support function of $K \in \mathcal{K}^n$, both with respect to the parameter defining the full system of parallel bodies of K with respect to E. In the classical case ($p = 1$), Sangwine-Yager [30, Lemma 4.9] stated that

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Big|_{\mu=\lambda} h(K_\mu, u) \geq h(K^*_\lambda, u) \geq h(E, u), \tag{1.7}$$

where $K^*_\lambda := (K_\lambda)^*$ denotes the form body of K_λ with respect to E.

In [15, Theorem 4.1] it is shown that in the case $1 \leq p < \infty$, if $E \in \mathcal{K}^n_0$ is regular and $K \in \mathcal{K}^n_{00}(E)$, then the function $\lambda \mapsto \mathrm{W}_{n-1}(K^p_\lambda; E)$ is differentiable, and its derivative can be computed by differentiating under the integral sign in the integral representation of $\mathrm{W}_{n-1}(K^p_\lambda; E)$, both with respect to the parameter λ.

We provide an improvement of the (best) lower bound of (1.7) in the case $1 \leq p < \infty$.

Theorem 1.5. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $\mathrm{r} = \mathrm{r}(K;E)$ and let $1 \le p < \infty$. Then for almost all $\lambda \in (-\mathrm{r}, 0]$ and every $u \in \mathbb{S}^{n-1}$,*

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Big|_{\mu=\lambda} h(K^p_\mu, u) \ge \frac{|\lambda|^{p-1} h\big((K^p_\lambda)^*, u\big)^p}{h(K^p_\lambda, u)^{p-1}}. \tag{1.8}$$

Geometric properties of p-inner parallel bodies, in general, and inner parallel bodies, in particular, can be already found in the literature (see e.g. [11, 15, 18, 19, 20, 22, 24, 25, 28, 29, 30]). In [17] and [30] several relations between the set of 0-extreme vectors of the inner parallel bodies of K and the set of 0-extreme vectors of K itself were established. We will make use of some of these results, in particular, of the arguments in their proofs, along the paper.

The integral formula (1.6) for $i = 0$ establishes that

$$\mathrm{vol}(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K, u)\, \mathrm{dS}(K[n-1], u).$$

However, since $\operatorname{supp} \mathrm{S}(K[n-1], \cdot) = \mathrm{cl}\,\mathcal{U}_0(K)$ for $K \in \mathcal{K}^n_n$ ([32, Theorem 4.5.3]), if $K = \mathrm{WS}(\psi)$ we also have

$$\mathrm{vol}(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \psi(u)\, \mathrm{dS}(K[n-1], u).$$

The latter suggests a strong link between Wulff shapes and 0-extreme vectors, and therefore we may expect to use the Wulff-shape structure in order to obtain properties of the boundary of convex bodies. The following proposition contains two statements in this spirit. They were proven by the second author in [27, Propositions 4.1.8 and 4.1.11]. We include the proofs in Section 2 for completeness.

Proposition 1.6. *Let $1 \le p < \infty$, and let $K, L, E \in \mathcal{K}^n_0$.*

i) *Then*

$$\mathcal{U}_0(K +_p L) \supseteq \mathcal{U}_0(K) \cup \mathcal{U}_0(L).$$

ii) *Let further $K \in \mathcal{K}^n_{00}(E)$ and $\mathrm{r} = \mathrm{r}(K;E)$. Then, for all $-\mathrm{r} < \lambda < 0$ we have*

$$\mathcal{U}_0(K^p_\lambda) \subseteq \mathcal{U}_0(K).$$

In the literature, we can find upper and lower bounds for the quermassintegrals, in particular for the volume, of K_λ or K^p_λ relative to E in terms of quermassintegrals of K, E and the form body of K (with respect to E), see [3, 24, 25, 31]. We establish new inequalities in the case $1 \le p < \infty$, by proving the following upper bound for the i-th quermassintegral of K^p_λ involving K^*_p, the so-called p-form body of K with respect to E, which is a generalization of the above defined form body (see Section 3 for the definition).

Theorem 1.7. *Let $E\in\mathcal{K}^n_0$, $K\in\mathcal{K}^n_{00}(E)$, $1\leq p<\infty$, $\mathrm{r}=\mathrm{r}(K;E)$, and let $i\in\{0,1,\dots,n-1\}$. Then, for all $\lambda\in[-\mathrm{r},0]$ we have that*

$$\begin{aligned}\alpha_p^{n-i}\mathrm{W}_i(K^p_\lambda;E)\leq \mathrm{W}_i(K;E)-|\lambda|\alpha_p\mathrm{V}(K[n-i-1],K^*_p,E[i])\\ -|\lambda|\sum_{j=1}^{n-i-1}\alpha_p^j\mathrm{V}(K^p_\lambda[j],K[n-i-j-1],K^*_p,E[i]),\end{aligned}$$

where we write, for the sake of brevity, $\alpha_p=2^{\frac{1}{p}-1}$.

The paper is organized as follows. In Section 2 we introduce the notion of 0-extreme vectors in detail, along with some considerations for the set of 0-extreme vectors of p-inner parallel bodies and the (classical) form body. Furthermore, for $K\in\mathcal{K}^n$, we investigate the set of 0-extreme vectors of two new convex bodies, closely linked to K and all its inner parallel bodies. In Section 3 we define the p-form body of a convex body containing the origin. This allows us to prove Theorem 1.5, as its proof relies strongly on properties of the p-form body, addressed also in this section. Finally, in Section 4 we prove Theorem 1.7, as well as other bounds for the relative (mixed) quermassintegrals of the p-inner parallel bodies.

2. 0-extreme vectors of convex bodies and related concepts

The set of 0-extreme normal vectors of a convex body $K\in\mathcal{K}^n$ plays an important role in the study of singularities of convex bodies (see e.g. [32, Section 2.2]), for they characterize, for instance, the regularity of convex bodies in terms of their 0-extreme vectors:

$$K\in\mathcal{K}^n \text{ is regular if and only if } \mathcal{U}_0(K)=\mathbb{S}^{n-1}. \tag{2.1}$$

The following characterization of 0-extreme vectors can be found in [30, Lemma 2.3] and will be useful throughout this section, being a fundamental tool to prove Propositions 1.6 and 2.4:

$$\begin{aligned}&u\in\mathcal{U}_0(K) \text{ if and only if for } u_1\neq u_2 \text{ in } \mathbb{S}^{n-1} \text{ such that}\\ &u=\alpha u_1+\beta u_2, \text{ with } \alpha,\beta>0, \text{ one has}\\ &\qquad h(K,u)<\alpha h(K,u_1)+\beta h(K,u_2).\end{aligned} \tag{2.2}$$

In 1978, Sangwine-Yager proved in [30, Lemma 4.5] that for $K,E\in\mathcal{K}^n_n$ with $\mathrm{r}=\mathrm{r}(K;E)$, one has

$$\mathcal{U}_0(K_\lambda)\subseteq\mathcal{U}_0(K) \quad \text{for all } -\mathrm{r}<\lambda<0. \tag{2.3}$$

And furthermore (see [30, Lemma 2.4]), that for $K,E\in\mathcal{K}^n$,

$$\mathcal{U}_0(K+L)\supseteq\mathcal{U}_0(K)\cup\mathcal{U}_0(L).$$

Notice that Proposition 1.6 is the natural extension of the above results to the case $1<p<\infty$. As mentioned in the introduction, we include its proof for completeness.

Proof of Proposition 1.6. The assertion in i) follows in a straightforward manner from (2.2). Indeed, let $u \in \mathcal{U}_0(K)$ and let $u_1, u_2 \in \mathbb{S}^{n-1}$, $u_1 \neq u_2$, be such that $u = \alpha u_1 + \beta u_2$, with $\alpha, \beta > 0$. Then, by (2.2) we have

$$h(K,u) < \alpha h(K,u_1) + \beta h(K,u_2).$$

For L, the subadditivity of the support function gives also

$$h(L,u) \leq \alpha h(L,u_1) + \beta h(L,u_2).$$

These inequalities, together with Minkowski's inequality for sums of real numbers (see e.g. [5, Theorem 9.5]), yield

$$\begin{aligned} h(K +_p L, u) &= \big(h(K,u)^p + h(L,u)^p\big)^{1/p} \\ &< \Big(\big[\alpha h(K,u_1) + \beta h(K,u_2)\big]^p + \big[\alpha h(L,u_1) + \beta h(L,u_2)\big]^p\Big)^{1/p} \\ &\leq \Big(\big(\alpha h(K,u_1)\big)^p + \big(\alpha h(L,u_1)\big)^p\Big)^{1/p} + \Big(\big(\beta h(K,u_2)\big)^p + \big(\beta h(L,u_2)\big)^p\Big)^{1/p} \\ &= \alpha\Big(h(K,u_1)^p + h(L,u_1)^p\Big)^{1/p} + \beta\Big(h(K,u_2)^p + h(L,u_2)^p\Big)^{1/p} \\ &= \alpha\, h(K +_p L, u_1) + \beta\, h(K +_p L, u_2). \end{aligned}$$

Then, (2.2) implies that $u \in \mathcal{U}_0(K +_p L)$, and thus $\mathcal{U}_0(K) \subseteq \mathcal{U}_0(K +_p L)$. Analogously, we get $\mathcal{U}_0(L) \subseteq \mathcal{U}_0(K +_p L)$, which concludes the proof of i).

Now we prove ii). Let $\lambda \in (-\mathrm{r}, 0)$ and let

$$u \in \mathcal{U}_0(K^p_\lambda) \subseteq \mathrm{cl}\,\mathcal{U}_0(K^p_\lambda) = \mathrm{supp}\,\mathrm{S}(K^p_\lambda[n-1], \cdot)$$

([32, Theorem 4.5.3 and (7.100)]). Then,

$$h(K^p_\lambda, u) = \big(h(K,u)^p - |\lambda|^p h(E,u)^p\big)^{1/p} \tag{2.4}$$

(cf. Remark 1.4). Let $u_1, u_2 \in \mathbb{S}^{n-1}$ and $\alpha, \beta > 0$ be such that $u = \alpha u_1 + \beta u_2$. By (1.1), (2.2) and (2.4), and taking into account the Minkowski inequality for sums of real numbers (see e.g. [5, Theorem 9.5]), we obtain that

$$\begin{aligned} h(K,u) &= \big(h(K^p_\lambda,u)^p + |\lambda|^p h(E,u)^p\big)^{1/p} \\ &< \Big(\big[\alpha h(K^p_\lambda,u_1) + \beta h(K^p_\lambda,u_2)\big]^p + |\lambda|^p\big[\alpha h(E,u_1) + \beta h(E,u_2)\big]^p\Big)^{1/p} \\ &\leq \Big(\big(\alpha h(K^p_\lambda,u_1)\big)^p + \big(\alpha|\lambda| h(E,u_1)\big)^p\Big)^{1/p} + \Big(\big(\beta h(K^p_\lambda,u_2)\big)^p + \big(\beta|\lambda| h(E,u_2)\big)^p\Big)^{1/p} \\ &= \alpha\Big(h(K^p_\lambda,u_1)^p + |\lambda|^p h(E,u_1)^p\Big)^{1/p} + \beta\Big(h(K^p_\lambda,u_2)^p + |\lambda|^p h(E,u_2)^p\Big)^{1/p} \\ &\leq \alpha h(K,u_1) + \beta h(K,u_2). \end{aligned}$$

In virtue of (2.2) we conclude that $u \in \mathcal{U}_0(K)$. □

Remark 2.1. *Proposition 1.6 implies that every p-inner parallel body of a polytope is also a polytope, for all* $1 \leq p < \infty$.

From (2.3) we immediately have that

$$\bigcup_{-\mathrm{r}<\lambda<0} \mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0(K). \tag{2.5}$$

In [23] it was shown that, for instance, if $K \in \mathcal{K}^n$ is a polytope, then equality holds in (2.5) when $E = B_n$. The following result shows that just assuming that $E \in \mathcal{K}^n_{(0)}$ is regular and strictly convex, then one can get equality in (2.5) but involving the closures of the sets of extreme vectors.

Proposition 2.2. *Let* $K \in \mathcal{K}^n_n$, *let* $E \in \mathcal{K}^n_{(0)}$ *be regular and strictly convex, and let* $\mathrm{r} = \mathrm{r}(K;E)$. *Then,*

$$\bigcup_{-\mathrm{r}<\lambda<0} \mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K).$$

Proof. Let $K \in \mathcal{K}^n_n$. Then $K_\lambda \in \mathcal{K}^n_n$ for $-\mathrm{r} < \lambda < 0$, and thus, by [32, Theorem 4.5.3], we have $\operatorname{supp} \mathrm{S}(K_\lambda[n-1],\cdot) = \mathrm{cl}\,\mathcal{U}_0(K_\lambda)$. We observe that K_λ converges to K if λ tends to 0 in the Hausdorff sense (for a precise definition and properties of the Hausdorff metric we refer to [32, Section 1.8]). Since the surface area measures are weakly convergent (see [21, Theorem 4.2]), we have that

$$S\big(K_\lambda[n-1],\cdot\big) \to S\big(K[n-1],\cdot\big)$$

weakly, as λ tends to zero. Thus,

$$\operatorname{supp}\big(S(K[n-1],\cdot\big) \subset \liminf \operatorname{supp}\big(S(K_\lambda[n-1],\cdot\big)$$

which implies that

$$\begin{aligned}\mathrm{cl}\,\mathcal{U}_0(K) &\subset \liminf \mathrm{cl}\,\mathcal{U}_0(K_\lambda)\\ &= \lim \mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\left(\bigcup_{-\mathrm{r}<\lambda<0} \mathrm{cl}\,\mathcal{U}_0(K_\lambda)\right).\end{aligned}$$

From (2.3) the claim follows:

$$\bigcup_{-\mathrm{r}<\lambda<0} \mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K). \qquad \square$$

2.1. **The Riemann-Minkowski integral of the inner sets.** Next we study some properties relating the 0-extreme vectors of the inner parallel bodies of a convex body and the so-called Riemann-Minkowski integral of the inner parallel bodies, which is defined as follows.

Definition 2.3. Let $K, E \in \mathcal{K}^n$ and let $-\mathrm{r} := -\mathrm{r}(K;E) \le a < b < \infty$. The *Riemann-Minkowski integral* of the family $\{K_\lambda\}_{\lambda\in(-\mathrm{r},0]}$ between a and b is the convex body $\int_a^b K_\lambda \,\mathrm{d}\lambda$ whose support function is given by

$$h\left(\int_a^b K_\lambda \,\mathrm{d}\lambda, u\right) = \int_a^b h(K_\lambda, u)\,\mathrm{d}\lambda \quad \text{for all } u \in \mathbb{S}^{n-1}.$$

Notice that $\{K_\lambda\}_{-\mathrm{r}<\lambda<0}$ is continuous in λ with respect to the Hausdorff metric in $\mathcal{K}^n$ ([28, Proposition 4.3]), and so the function $\lambda \mapsto h(K_\lambda, u)$ is integrable for all $u \in \mathbb{S}^{n-1}$. The Riemann-Minkowski integral of a bounded family of convex bodies was introduced by Dinghas in [7], and was treated in [8, 10, 30] as a useful tool to obtain geometric inequalities.

A relation between both, 0-extreme vectors of the inner parallel bodies of K with respect to E, and 0-extreme vectors of the Riemann-Minkowski integral of $\{K_\lambda\}_{\lambda\in[-\mathrm{r},0]}$ is stated in the following result.

Proposition 2.4. *Let $K, E \in \mathcal{K}^n$, $\mathrm{r} = \mathrm{r}(K;E)$ and $-\mathrm{r} \le a < b \le 0$. Then,*

$$\bigcup_{a\le\lambda<b} \mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0\left(\int_a^b K_\lambda \,\mathrm{d}\lambda\right).$$

Proof. Let $\lambda_0 \in [a, b)$, and let $u \in \mathcal{U}_0(K_{\lambda_0})$. Using (2.3) we obtain that $\mathcal{U}_0(K_\eta) \subseteq \mathcal{U}_0(K_\xi)$ for $-\mathrm{r} \le \eta \le \xi \le 0$. Then, $u \in \mathcal{U}_0(K_\lambda)$ for all $\lambda \in [\lambda_0, b)$. Let $u_1 \ne u_2$ in $\mathbb{S}^{n-1}$ and $\alpha, \beta > 0$ be such that $u = \alpha u_1 + \beta u_2$. Then, by (2.2) and using the sublinearity of the support function,

$$\begin{aligned} h\left(\int_a^b K_\lambda \mathrm{d}\lambda, u\right) &= \int_a^b h(K_\lambda, u)\,\mathrm{d}\lambda \\ &= \int_a^{\lambda_0} h(K_\lambda, u)\,\mathrm{d}\lambda + \int_{\lambda_0}^b h(K_\lambda, u)\,\mathrm{d}\lambda \\ &< \alpha \int_a^b h(K_\lambda, u_1)\,\mathrm{d}\lambda + \beta \int_a^b h(K_\lambda, u_2)\,\mathrm{d}\lambda. \end{aligned}$$

Therefore, $u \in \mathcal{U}_0(\int_a^b K_\lambda \,\mathrm{d}\lambda)$. □

As a consequence of the latter and Proposition 2.2, we obtain a condition for the regularity of the Riemann-Minkowski integral of $\{K_\lambda\}_\lambda$.

Corollary 2.5. *Let $K, E \in \mathcal{K}^n$ be regular, with E strictly convex, and let $\mathrm{r} = \mathrm{r}(K;E)$. Then $\mathrm{cl}\,\mathcal{U}_0\left(\int_{-\mathrm{r}}^0 K_\lambda \,\mathrm{d}\lambda\right) = \mathbb{S}^{n-1}$.*

Proof. Since K is regular, we have that $\mathrm{cl}\,\mathcal{U}_0(K) = \mathbb{S}^{n-1}$. Propositions 2.2 and 2.4 imply that

$$\begin{aligned} \mathbb{S}^{n-1} = \mathrm{cl}\,\mathcal{U}_0(K) &= \bigcup_{-\mathrm{r}<\lambda<0} \mathrm{cl}\,\mathcal{U}_0(K_\lambda) \\ &\subseteq \mathrm{cl}\left(\bigcup_{-\mathrm{r}<\lambda<0} \mathcal{U}_0(K_\lambda)\right) \subseteq \mathrm{cl}\,\mathcal{U}_0\left(\int_{-\mathrm{r}}^0 K_\lambda \,\mathrm{d}\lambda\right). \end{aligned}$$

Therefore, $\mathrm{cl}\,\mathcal{U}_0\big(\int_{-\mathrm{r}}^0 K_\lambda \,\mathrm{d}\lambda\big) = \mathbb{S}^{n-1}$. □

The statements so far do not involve the form body, which happens to be fundamental for our main results. Next, we will deal with 0-extreme vectors of the form body in connection to the previous results. Sangwine-Yager proved in [30, Lemma 4.6] that for full-dimensional convex bodies $K, E \in \mathcal{K}^n_n$, the inclusion

$$\mathcal{U}_0(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_0(K) \tag{2.6}$$

holds true. In [17, Lemma 2.1] it is shown that if $E \in \mathcal{K}^n_n$ is regular, then equality holds in (2.6) for all $K \in \mathcal{K}^n_n$.

Directing our attention to the convex body $\int_{-\mathrm{r}}^0 K^*_\lambda \,\mathrm{d}\lambda$, Sangwine-Yager established in [30, Lemma 3.2] that for $E, K \in \mathcal{K}^n$, with $0 \in \operatorname{int} E$ and $\mathrm{r} = \mathrm{r}(K; E)$, the following relation:

$$K_{-\mathrm{r}} + \int_{-\mathrm{r}}^0 K^*_\lambda \,\mathrm{d}\lambda \subseteq K. \tag{2.7}$$

Using the fact that, if $E \in \mathcal{K}^n_0$ is regular, then $\mathcal{U}_0(K^*) = \mathrm{cl}\,\mathcal{U}_0(K)$ [17, Lemma 2.1], we get, similarly to Corollary 2.5, that for $K \in \mathcal{K}^n$, $E \in \mathcal{K}^n_0$ regular, $\mathrm{r} = \mathrm{r}(K; E)$ and $-\mathrm{r} \le a < b \le 0$,

$$\bigcup_{a<\lambda<b} \mathcal{U}_0\left(K^*_\lambda\right) \subseteq \mathrm{cl}\,\mathcal{U}_0\left(\int_a^b K^*_\lambda \,\mathrm{d}\lambda\right).$$

The latter inclusion allows us to obtain an improvement of (2.5) in particular cases. For instance, if $E \in \mathcal{K}^n_0$ is regular and equality holds in (2.7), then we have the following relation, which improves (2.5):

$$\mathcal{U}_0(K) \supseteq \mathcal{U}_0\left(\int_{-\mathrm{r}}^0 K^*_\lambda \,\mathrm{d}\lambda\right) \supseteq \bigcup_{-\mathrm{r}<\lambda<0} \mathcal{U}_0\left(K^*_\lambda\right) \supseteq \bigcup_{-\mathrm{r}<\lambda<0} \mathcal{U}_0(K_\lambda).$$

2.2. **A general convex body associated to a convex set and a positive function.** Next we introduce a new convex body associated to a convex body $K \in \mathcal{K}^n_n$ and a positive function ψ over $\mathbb{S}^{n-1}$, which includes the relative form body and the inner parallel bodies as particular cases. Let $K \in \mathcal{K}^n_n$ and $\psi : \mathbb{S}^{n-1} \longrightarrow (0, \infty)$ be a (not necessarily continuous) function with $\inf_{u \in \mathcal{U}_0(K)} \psi(u) > 0$. Let us define the following set:

$$K_\psi := \mathrm{WS}(\mathcal{U}_0(K), \psi).$$

Notice that, as $\mathrm{lin}\,\mathcal{U}_0(K) = \mathbb{R}^n$, because K is full-dimensional, $K_\psi \in \mathcal{K}^n$. Moreover, $0 \in \operatorname{int} K_\psi$, since $\inf_{u \in \mathcal{U}_0(K)} \psi(u) > 0$. Hence, $K_\psi \in \mathcal{K}^n_{(0)}$. Observe that if $E \in \mathcal{K}^n_{(0)}$, then $K_{h(E,\cdot)} = K^*$.

The following auxiliary result will allow us to obtain the analogue of (2.6) for K_ψ (Proposition 2.7). Although it seems to have been used already in the literature, we include a short proof for completeness.

Lemma 2.6. *Let $K \in \mathcal{K}^n$ and let $\psi : \mathbb{S}^{n-1} \longrightarrow (0, \infty)$ be a continuous function. Then,*

$$\mathrm{cl}\left\{\frac{u}{\psi(u)} : u \in \mathcal{U}_0(K)\right\} = \left\{\frac{u}{\psi(u)} : u \in \mathrm{cl}\,\mathcal{U}_0(K)\right\}. \tag{2.8}$$

Proof. Let $x \in \operatorname{cl}\left\{\frac{u}{\psi(u)} : u \in \mathcal{U}_0(K)\right\}$. Then, we find a sequence of 0-extreme vectors $\{u_j\}_{j=1}^{\infty} \subseteq \mathcal{U}_0(K)$ such that $\lim_j \frac{u_j}{\psi(u_j)} = x$. Since $\mathcal{U}_0(K) \subseteq \mathbb{S}^{n-1}$ and $\mathbb{S}^{n-1}$ is compact, we may assume that $\lim_j u_j = \widetilde{u} \in \operatorname{cl}\mathcal{U}_0(K)$. Since ψ is continuous, we get that $x = \frac{\lim_j u_j}{\lim_j \psi(u_j)} = \frac{\widetilde{u}}{\psi(\widetilde{u})}$ for $\widetilde{u} \in \operatorname{cl}\mathcal{U}_0(K)$.

Conversely, let $x = \frac{u}{\psi(u)}$, with $u \in \operatorname{cl}\mathcal{U}_0(K)$. Then, there exists a sequence $\{u_j\}_{j=1}^{\infty} \subseteq \mathcal{U}_0(K)$ with $\lim_j u_j = u$. Since ψ is continuous, we have that

$$x = \frac{\lim_j u_j}{\psi\left(\lim_j u_j\right)} = \lim_j \frac{u_j}{\psi(u_j)} \in \operatorname{cl}\left\{\frac{u}{\psi(u)} : u \in \mathcal{U}_0(K)\right\}. \qquad \square$$

The proof of the following result follows the lines of the proof of (2.6) in [30, Lemma 4.6], now in a slightly more general setting, taking into account Lemma 2.6. We include the complete argument here for completeness. Recall that, for any subset $A \subseteq \mathbb{R}^n$, the *dual* or *polar set* of A, denoted by A°, is defined by

$$A^\circ := \{x \in \mathbb{R}^n : \langle x, a\rangle \leq 1 \text{ for all } a \in A\}$$

and, as usual in the literature, we will write $A^{\circ\circ} = (A^\circ)^\circ$. The different properties of the dual set that will be used throughout the proof can be found in [33].

Proposition 2.7. *Let $K \in \mathcal{K}_n^n$ and let $\psi : \mathbb{S}^{n-1} \longrightarrow (0,\infty)$ be a continuous function. Then, $\mathcal{U}_0(K_\psi) \subseteq \operatorname{cl}\mathcal{U}_0(K)$.*

Proof. For all $u \in \mathcal{U}_0(K)$, we define

$$X_u = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq \psi(u)\} = \left\{\frac{u}{\psi(u)}\right\}^\circ.$$

Thus, $K_\psi = \bigcap_{u\in\mathcal{U}_0(K)} X_u$. Let us define also

$$Y_u = X_u^\circ = \left\{\frac{u}{\psi(u)}\right\}^{\circ\circ} = \operatorname{cl}\operatorname{conv}\left(\left\{\frac{u}{\psi(u)}\right\} \cup \{0\}\right) = \left\{\frac{\theta u}{\psi(u)} : 0 \leq \theta \leq 1\right\}.$$

Then,

$$Y_u^\circ = \left\{\frac{u}{\psi(u)}\right\}^{\circ\circ\circ} = \left\{\frac{u}{\psi(u)}\right\}^{\circ} = X_u,$$

and thus, we have that

$$\left(\bigcup_{u\in\mathcal{U}_0(K)} Y_u\right)^\circ = \bigcap_{u\in\mathcal{U}_0(K)} Y_u^\circ = \bigcap_{u\in\mathcal{U}_0(K)} X_u = K_\psi.$$

Therefore, by Lemma 2.6, and taking into account that $0 \in Y_u$ for all $u \in \mathcal{U}_0(K)$, we deduce that

$$K_\psi^\circ = \left(\bigcup_{u\in\mathcal{U}_0(K)} Y_u\right)^{\circ\circ} = \operatorname{conv}\left\{\frac{u}{\psi(u)} : u \in \operatorname{cl}\mathcal{U}_0(K)\right\}.$$

Thus, if y is an extreme boundary point of K_ψ°, then $y = \frac{u}{\psi(u)}$ for some $u \in \operatorname{cl}\mathcal{U}_0(K)$.

Now, let $u_0 \in \mathcal{U}_0(K_\psi)$. By duality, we find that $y = \frac{u_0}{h(K_\psi, u_0)}$ is an extreme point of K_ψ° (see [30, Lemma 2.2]). Notice that $h(K_\psi, u_0) > 0$ because $0 \in \operatorname{int} K_\psi$. Then, $u_0 = \frac{h(K_\psi, u_0)}{\psi(u)} u$ for some $u \in \operatorname{cl}\mathcal{U}_0(K)$. Since $h(K_\psi, u_0), \psi(u) > 0$ and $u, u_0 \in \mathbb{S}^{n-1}$, it must be $h(K_\psi, u_0) = \psi(u)$, from which $u_0 = u \in \operatorname{cl}\mathcal{U}_0(K)$. □

Notice that if ψ is not a continuous function, the result of Lemma 2.6 can fail, as the following example shows.

Example 2.8. We are going to see that there exist $K \in \mathcal{K}^3$ and a non-continuous function $\psi : \mathbb{S}^2 \longrightarrow (0, \infty)$ with $\inf_{u \in \mathcal{U}_0(K)} \psi(u) > 0$ such that (2.8) does not hold.

Let $H_i^- = \{(x, y, z)^\intercal \in \mathbb{R}^3 : y + (-1)^i z \leq 1\}$, $i = 1, 2$, and we consider the truncated cylinder $K = H_1^- \cap H_2^- \cap \{(x, y, z)^\intercal \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ (see Figure 1). It is easy to see that $\mathrm{e}_2 = (0, 1, 0)^\intercal \in \operatorname{cl}\mathcal{U}_0(K) \backslash \mathcal{U}_0(K)$; indeed,

$$\mathcal{U}_0(K) = \{(\sin\theta, \cos\theta, 0)^\intercal : \theta \in (0, 2\pi)\} \cup \left\{\left(0, \frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)^\intercal\right\}.$$

We define $\psi : \mathbb{S}^2 \longrightarrow (0, \infty)$ by $\psi(u) = 1$ if $u \neq \mathrm{e}_2$ and $\psi(\mathrm{e}_2) = 1/2$. Clearly, ψ is not continuous at e_2. On the one hand, we have that

$$\operatorname{cl}\left\{\frac{u}{\psi(u)} : u \in \mathcal{U}_0(K)\right\} = \{(\sin\theta, \cos\theta, 0)^\intercal : \theta \in [0, 2\pi]\} \cup \left\{\left(0, \frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)^\intercal\right\}.$$

On the other hand, $\left\{\frac{u}{\psi(u)} : u \in \operatorname{cl}\mathcal{U}_0(K)\right\} = \mathcal{U}_0(K) \cup \{2\mathrm{e}_2\}$, and thus (2.8) fails to be true.

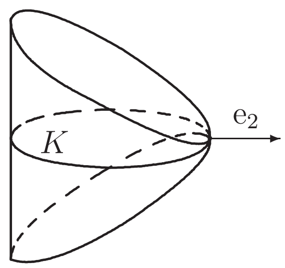

FIGURE 1. The truncated cylinder K.

3. On p-decompositions contained in a convex body

From the definition of Minkowski difference, we have that K_λ is the largest convex body such that $K_\lambda + |\lambda| E \subseteq K$, for all $-\mathrm{r}(K; E) \leq \lambda \leq 0$. Moreover, it is known that

$$K_\lambda + |\lambda| K^* \subseteq K, \tag{3.1}$$

for all $-\mathrm{r}(K;E) \leq \lambda \leq 0$ (see [30, Lemma 4.8]). Relation (3.1) allows us to approach the convex body K by means of a Minkowski sum inside K, better than using E, by using K^* (which contains E) instead. When $K, E \in \mathcal{K}^n_0$, and E is regular, equality conditions of (3.1) were given in [18, Theorem 2.2].

For $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leq p < \infty$ and $\mathrm{r} = \mathrm{r}(K;E)$, we obtain from the definition of p-difference that for all $-\mathrm{r} \leq \lambda \leq 0$, K^p_λ is the largest convex body such that $K^p_\lambda +_p |\lambda| E \subseteq K$ (see [28]). Then it is a natural question to wonder which is the largest convex body $L \supseteq E$ satisfying $K^p_\lambda +_p |\lambda| L \subseteq K$, for all $-\mathrm{r} \leq \lambda \leq 0$. Taking into account this idea, we introduce the following convex body.

Definition 3.1. Let $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leq p < \infty$ and let $\mathrm{r} = \mathrm{r}(K;E)$. The (relative) *p-form body* of K *with respect to* E, denoted by K^*_p, is the set $K^*_p := \mathrm{WS}\bigl(\mathcal{U}_0(K), f^K_p\bigr)$, where $f^K_p : \mathbb{S}^{n-1} \longrightarrow [0,\infty)$ is the function given by

$$f^K_p(u) := \inf_{-\mathrm{r} \leq \lambda < 0} \frac{1}{|\lambda|} \bigl(h(K,u)^p - h(K^p_\lambda, u)^p\bigr)^{1/p}.$$

As in the case of the (classical) form body, K^*_p depends on the gauge body E, but we do not write this dependence explicitly for the sake of brevity. Note that for any $\lambda \in [-\mathrm{r}(K;E), 0]$, $|\lambda| f^K_p(u) \leq \bigl(h(K,u)^p - h(K^p_\lambda, u)\bigr)^{1/p}$, and by (1.3), $f^K_p(u) \geq 0$ for all $u \in \mathbb{S}^{n-1}$. Indeed, $f^K_p(u) \geq h(E,u)$ for all $u \in \mathcal{U}_0(K)$ (cf. (1.1) and (1.3)), which implies, in virtue of the dual of the Krein-Milman theorem, that $E \subseteq K^*_p$. In particular, K^*_p cannot be lower-dimensional if $E \in \mathcal{K}^n_n$. The p-form body satisfies analogous properties to those of the classical form body. We start by proving the following result.

Proposition 3.2. *Let* $E \in \mathcal{K}^n_0$*, and let* $K \in \mathcal{K}^n_{00}(E)$*. Let further* $1 \leq p < \infty$*, and let* $\mathrm{r} = \mathrm{r}(K;E)$*. Then:*

i) $K^*_p \in \mathcal{K}^n_0$.
ii) $K^* \subseteq K^*_p$.
iii) $K^p_\lambda +_p |\lambda| K^*_p \subseteq K$*, for all* $-\mathrm{r} \leq \lambda \leq 0$.

Proof. We start by proving ii). Let $x \in K^*$. Then, $\langle x, u\rangle \leq h(E,u)$ for all $u \in \mathcal{U}_0(K)$. As we noticed before, we have that $f^K_p(u) \geq h(E,u)$ for all $u \in \mathcal{U}_0(K)$. Therefore, $\langle x, u\rangle \leq h(E,u) \leq f^K_p(u)$ for all $u \in \mathcal{U}_0(K)$, and hence $x \in K^*_p$.

iii) If $\lambda = 0$ the result trivially holds. So, let $-\mathrm{r} \leq \lambda < 0$ and take $u \in \mathcal{U}_0(K)$. Then we have (cf. Remark 1.4 and (1.3))

$$h\bigl(K^p_\lambda +_p |\lambda| K^*_p, u\bigr)^p \leq h(K^p_\lambda, u)^p + |\lambda|^p f^K_p(u)^p \leq h(K,u)^p.$$

Thus, $h\bigl(K^p_\lambda +_p |\lambda| K^*_p, u\bigr) \leq h(K,u)$ for all $u \in \mathcal{U}_0(K)$, and therefore, the dual of the Krein-Milman theorem ensures that $K^p_\lambda +_p |\lambda| K^*_p \subseteq K$.

Finally we prove i). By construction, K_p^* is an intersection of closed halfspaces, and thus it is closed and convex. Moreover, by iii) we have that $K_p^* \subseteq \frac{1}{\mathrm{r}} K$, and hence K_p^* is bounded. Since $0 \in K_p^*$, we deduce that $K_p^* \in \mathcal{K}_0^n$. □

The following result shows that the p-form body K_p^*, as its more classical counterpart, is invariant under dilations of the body K.

Lemma 3.3. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \le p < \infty$ and $\theta > 0$. Then,*

$$(\theta K)_p^* = K_p^*.$$

Proof. Let $u \in \mathcal{U}_0(\theta K)$. By Proposition 1.2 v) and taking into account that the inradius is positively homogeneous, namely, $\mathrm{r}(\theta K; E) = \theta \mathrm{r}(K; E)$, we obtain that $f_p^{\theta K}(u) = f_p^K(u)$. Since $\mathcal{U}_0(\theta K) = \mathcal{U}_0(K)$ for any $\theta > 0$, we get the result. □

Remark 3.4. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \le p < \infty$ and let $\mathrm{r} = \mathrm{r}(K; E)$. Then,*

$$K_{-\mathrm{r}}^p +_p (\mathrm{r} +_p \lambda) K_p^* \subseteq K_\lambda^p$$

for all $-\mathrm{r} \le \lambda \le 0$. Indeed, since $E \subseteq K_p^*$, using Propositions 1.2 and 3.2 we obtain that $\left(K_{-\mathrm{r}}^p +_p (\mathrm{r} +_p \lambda) K_p^*\right) +_p |\lambda| E \subseteq K_{-\mathrm{r}}^p +_p \mathrm{r} K_p^* \subseteq K$, from what follows that $K_{-\mathrm{r}}^p +_p (\mathrm{r} +_p \lambda) K_p^* \subseteq K_\lambda^p$.

It is not difficult to see that the (classical) form body of K with respect to E is the largest convex body such that, for all $-\mathrm{r} \le \lambda \le 0$

$$K \sim |\lambda| E = K \sim |\lambda| K^*. \tag{3.2}$$

The latter inequality says that the geometry of K^* describes, in some sense, the geometry of the inner parallel bodies. We are going to see that the p-form body verifies an analog property to (3.2).

Lemma 3.5. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \le p < \infty$ and let $\mathrm{r} = \mathrm{r}(K; E)$. Then, for all $-\mathrm{r} \le \lambda \le 0$,*

$$K \sim_p |\lambda| E = K \sim_p |\lambda| K_p^*. \tag{3.3}$$

Proof. From Proposition 3.2 iii) we have that

$$K_\lambda^p = K \sim_p |\lambda| E \subseteq K \sim_p |\lambda| K_p^*.$$

The reverse inclusion follows from the fact that $E \subseteq K^* \subseteq K_p^*$ (see Proposition 3.2 ii)). □

The following result shows that the 1-form body coincides with the (classical) form body when the gauge body E is regular and strictly convex.

Corollary 3.6. *Let $K \in \mathcal{K}_n^n$ and let $E \in \mathcal{K}_{(0)}^n$ be regular and strictly convex. Then, $K_1^* = K^*$.*

Proof. Let $\mathrm{r} = \mathrm{r}(K;E)$. We know that $\mathrm{cl}\,\mathcal{U}_0(K) = \bigcup_{-\mathrm{r}<\lambda<0} \mathrm{cl}\,\mathcal{U}_0(K_\lambda)$ (see Proposition 2.2). So, given $u \in \mathrm{cl}\,\mathcal{U}_0(K)$, there exists $\lambda \in (-\mathrm{r}, 0)$ such that $u \in \mathrm{cl}\,\mathcal{U}_0(K_\lambda)$. Now, if $u \in \mathcal{U}_0(K_\lambda)$, then $h(K_\lambda, u) = h(K, u) - |\lambda| h(E, u)$ (see [30, Lemma 4.4]) and thus, if $u \in \mathrm{cl}\,\mathcal{U}_0(K)$, we also have $h(K_\lambda, u) = h(K, u) - |\lambda| h(E, u)$ by the continuity of the support function, and the continuity of the full system of parallel bodies $\{K_\lambda\}_{\lambda\in[-\mathrm{r},0]}$ in λ (see [28, Proposition 4.3]). Hence, $f_1^K(u) = h(E, u)$ for all $u \in \mathrm{cl}\,\mathcal{U}_0(K)$. Therefore,

$$\begin{aligned} K_1^* &= \bigcap_{u\in\mathcal{U}_0(K)} \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \le h(E, u)\bigr\} \\ &= \bigcap_{u\in \mathrm{cl}\,\mathcal{U}_0(K)} \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \le h(E, u)\bigr\} = K^*. \end{aligned}$$ □

Next we prove some properties of the p-form body and the p-kernel of the so-called tangential bodies. This kind of bodies can be defined in several equivalent ways; here we will use the following one: a convex body $K \in \mathcal{K}^n$ containing a convex body $E \in \mathcal{K}^n$ is called a *tangential body* of E, if through each boundary point of K there exists a support hyperplane to K that also supports E. We notice that if K is a tangential body of E, then $\mathrm{r}(K;E) = 1$. For instance, the n-dimensional cube $[-1, 1]^n$ is a tangential body of $E = B_n$. For an exhaustive study of the more generally defined p-tangential bodies, we refer to [32, Section 2.2 and p. 149]. Tangential bodies happen to be intrinsically connected to their p-inner parallel bodies, as they are characterized as the only convex bodies such that their p-inner parallel bodies are homothetic copies of themselves. We refer the reader to [32, Lemma 3.1.14] for the case $p = 1$, and [28, Theorem 4.2] for $p > 1$. We include the precise statement for completeness.

Theorem 3.7 ([32, Lemma 3.1.14], [28, Proposition 4.5 and Theorem 4.2])**.** *Let $K, E \in \mathcal{K}_0^n$ with $E \subseteq K$ and $\mathrm{r}(K;E) = 1$. Let $1 \le p < \infty$.*

- i) *If $K_\lambda^p = \theta K$ for some $\theta \in [0, 1]$ and $\lambda \in [-1, 0]$, then $\theta = \bigl(1-|\lambda|^p\bigr)^{1/p}$ and $K_{-(1-\theta)} = \theta K$.*
- ii) *Let $K, E \in \mathcal{K}_0^n$, $\mathrm{int}\,E \neq \emptyset$, with $E \subseteq K$ and $\mathrm{r}(K;E) = 1$. Let $1 \le p < \infty$. Then K is a tangential body of E if and only if K_λ^p is homothetic to K for all $\lambda \in [-1, 0]$.*

We observe that if K is a tangential body of E, then $h(K, u) = h(E, u)$ for all $u \in \mathcal{U}_0(K)$ by definition, and therefore, $K^* = K$. We prove a similar property for the p-form body of tangential bodies in the following result.

Lemma 3.8. *Let $E \in \mathcal{K}_0^n$ with $\mathrm{int}\,E \neq \emptyset$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. Then K is a tangential body of E if and only if $K_p^* = K$.*

Proof. We assume first that K is a tangential body of E. Then, Theorem 3.7 yields that $K_\lambda^p = \bigl(1 - |\lambda|^p\bigr)^{1/p} K$ for all $-1 \le \lambda \le 0$ and thus,

$$f_p^K(u) = \inf_{-1\le\lambda<0} \frac{1}{|\lambda|}\bigl(h(K, u)^p - h(K_\lambda^p, u)^p\bigr)^{1/p} = h(K, u)$$

for all $u\in\mathcal{U}_0(K)$. Therefore, $K_p^*=K$.

Reciprocally, let $K=K_p^*$. Using Remark 1.4 we have $h(K_p^*,u)=f_p^K(u)$ for all $u\in\mathcal{U}_0(K_p^*)=\mathcal{U}_0(K)$, and thus

$$h(K,u)=h(K_p^*,u)=f_p^K(u)$$

for all $u\in\mathcal{U}_0(K)$. Then,

$$h(K,u)^p\leq\frac{1}{|\lambda|^p}\bigl[h(K,u)^p-h(K_\lambda^p,u)^p\bigr]$$

for any $-\mathrm{r}(K;E)\leq\lambda\leq 0$ and $u\in\mathcal{U}_0(K)$ and hence

$$h(K_\lambda^p,u)\leq\bigl(1-|\lambda|^p\bigr)^{1/p}h(K,u).$$

Since $\mathcal{U}_0(K_\lambda^p)\subseteq\mathcal{U}_0(K)$, we can conclude that $K_\lambda^p\subseteq(1-|\lambda|^p)^{1/p}K$. Observing now that

$$\bigl(1-|\lambda|^p\bigr)^{1/p}K+_p|\lambda|E\subseteq\bigl(1-|\lambda|^p\bigr)^{1/p}K+_p|\lambda|K_p^*=\bigl(1-|\lambda|^p\bigr)^{1/p}K+_p|\lambda|K=K,$$

we get that $\bigl(1-|\lambda|^p\bigr)^{1/p}K=K_\lambda^p$. Theorem 3.7 yields the result. □

We notice that following a similar argument, it can be proved that K is a dilation of a tangential body of a E if and only if $K_p^*=K$.

Remark 3.9. *Let $E\in\mathcal{K}_0^n$ with $\operatorname{int}E\neq\emptyset$, let $K\in\mathcal{K}_{00}^n(E)$ be a tangential body of E and let $1\leq p<\infty$. Since $K_\lambda^p=\theta_\lambda K$ for all $-1\leq\lambda\leq 0$, with $\theta_\lambda=\bigl(1-|\lambda|^p\bigr)^{1/p}$, Lemmas 3.3 and 3.8 imply that $(K_\lambda^p)_p^*=(\theta_\lambda K)_p^*=K_p^*=K$ for all $-1\leq\lambda\leq 0$.*

On the other hand, from Lemma 3.8 and Proposition 3.2 iii) we have that $K_{-1}^p+_pK=K_{-1}^p+_pK_p^*\subseteq K$, which implies that $\ker_p(K;E)=K_{-1}^p=\{0\}$.

Heuristically, since the derivative of the support function of K_λ is greater than the support function of the form body K_λ^* (see (1.7)), one could think that the support function of the form body controls, in some sense, the rate of change of the support function of the inner parallel bodies. The spirit of Theorem 1.5 is to prove an analog of this control in the case of p-parallel bodies. Next we show the proof of this result.

Proof of Theorem 1.5. We know from Proposition 3.2 iii) that if $E\in\mathcal{K}_0^n$, $L\in\mathcal{K}_{00}^n(E)$, $\mathrm{r}_L=\mathrm{r}(L;E)$ and $1\leq p<\infty$, then

$$L_\rho^p+_p|\rho|L_p^*\subseteq L,\quad\text{for all }-\mathrm{r}_L\leq\rho\leq 0.\tag{3.4}$$

Hence, for $K\in\mathcal{K}_{00}^n(E)$ with $\mathrm{r}=\mathrm{r}(K;E)$ and $-\mathrm{r}\leq\lambda<\lambda+\varepsilon<0$, we have that $K_{\lambda+\varepsilon}^p\in\mathcal{K}_{00}^n(E)$. Thus, (3.4) and Proposition 1.2 iv) imply that

$$K_\lambda^p+_p\mu(\lambda,\varepsilon)(K_{\lambda+\varepsilon}^p)_p^*\subseteq K_{\lambda+\varepsilon}^p,$$

where $\mu(\lambda,\varepsilon)$ is the positive real number satisfying $\lambda+\varepsilon=\lambda+_p\mu(\lambda,\varepsilon)$, namely, $\mu(\lambda,\varepsilon)=\bigl(|\lambda|^p-(|\lambda|-\varepsilon)^p\bigr)^{1/p}$. From this and Proposition 3.2 ii) we obtain that

$$K_\lambda^p+_p\mu(\lambda,\varepsilon)(K_{\lambda+\varepsilon}^p)^*\subseteq K_{\lambda+\varepsilon}^p.\tag{3.5}$$

Following the same steps of the proof of [15, Theorem 1.4], and using (3.5) and the continuity of the full system $\{K_\lambda^p\}_\lambda$ in λ (see [28, Proposition 4.3]), we obtain that for $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \le p < \infty$, the inequality

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Big|_{\mu=\lambda} h(K_\mu^p, u) \ge \frac{|\lambda|^{p-1}}{h(K_\lambda^p, u)^{p-1}} \lim_{\varepsilon\to 0^+} h\left((K_{\lambda+\varepsilon}^p)^*, u\right)^p = \frac{|\lambda|^{p-1} h\left((K_\lambda^p)^*, u\right)^p}{h(K_\lambda^p, u)^{p-1}},$$

holds for almost all $\lambda \in [-\mathrm{r}, 0]$, where the equality $lim_{\varepsilon\to 0^+} h\big((K_{\lambda+\varepsilon}^p)^*, u\big) = h\big((K_\lambda^p)^*, u\big)$ is guaranteed by the continuity of $\{K_\lambda^p\}_\lambda$ in λ and the proof of [30, Lemma 3.1]. Thus, we obtain (1.8). □

If we express the Riemann-Minkowski integral $\int_{-\mathrm{r}}^0 K_\lambda^* \,\mathrm{d}\lambda$ as a Wulff-shape, namely, $\int_{-\mathrm{r}}^0 K_\lambda^* \,\mathrm{d}\lambda = \mathrm{WS}(\phi)$ for $\phi(u) := \int_{-\mathrm{r}}^0 h(K_t^*, u)\,\mathrm{d}t$, then (2.7) can be rewritten as

$$K_{-\mathrm{r}} + \mathrm{WS}(\phi) \subseteq K. \tag{3.6}$$

The following result establishes an analogous relation to (3.6) in the case $1 < p < \infty$. Before showing the statement we introduce the auxiliary function $\phi_{p,\lambda} : \mathbb{S}^{n-1} \longrightarrow [0, \infty)$ given by

$$\phi_{p,\lambda}(u) := \left(p \int_\lambda^0 |t|^{p-1} h\left((K_t^p)^*, u\right)^p \,\mathrm{d}t\right)^{1/p}.$$

Corollary 3.10. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $\mathrm{r} = \mathrm{r}(K; E)$ and $1 \le p < \infty$, and let $-\mathrm{r} \le \lambda \le 0$. Then,*

$$K_\lambda^p +_p \mathrm{WS}(\phi_{p,\lambda}) \subseteq K. \tag{3.7}$$

Proof. If we integrate the inequality in Theorem 1.5, i.e., (1.8) in $[\lambda, 0]$, we obtain that

$$\frac{1}{p}\left[h(K, u)^p - h(K_\lambda^p, u)^p\right] \ge \int_\lambda^0 |t|^{p-1} h\left((K_t^p)^*, u\right)^p \,\mathrm{d}t,$$

i.e., $h(K, u)^p \ge h(K_\lambda^p, u)^p + \phi_{p,\lambda}(u)^p$ for all $u \in \mathbb{S}^{n-1}$. From here we get immediately the result using the support functions of the involved convex bodies and Remark 1.4. □

Notice that $\phi_{1,-\mathrm{r}} = \phi$, and thus we can recover (2.7) from Corollary 3.10.

4. Geometric inequalities for the family of p-parallel bodies

This section is devoted to prove inequalities for different quermassintegrals of a given convex body, K, and its p-parallel bodies, with respect to a fixed gauge body E. First, we recall the definition of mixed quermassintegrals introduced by Lutwak in [26], as well as their integral representation.

Theorem 4.1. ([32, Theorem 9.1.1][26, Lemma 3.2]) *Let $K, L \in \mathcal{K}^n_{(0)}$, $E \in \mathcal{K}^n_n$ and $p \geq 1$. Then, for all $i \in \{0, 1, \ldots, n-1\}$,*

$$\begin{aligned}\frac{n-i}{p}\mathrm{W}_{p,i}(K, L; E) &:= \lim_{\varepsilon \to 0^+} \frac{\mathrm{W}_i(K +_p \varepsilon^{1/p} L; E) - \mathrm{W}_i(K; E)}{\varepsilon} \\ &= \frac{n-i}{p}\frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L, u)^p h(K, u)^{1-p}\,\mathrm{dS}(K[n-i-1], E[i], u).\end{aligned} \tag{4.1}$$

From Remark 3.4 we get that $(\mathrm{r} +_p \lambda)K_p^* \subseteq K_\lambda^p$. In the following, we will use the weaker inclusion $(\mathrm{r} +_p \lambda)E \subseteq K_\lambda^p$, in order to deal with the (mixed) quermassintegrals of the involved convex bodies.

The following result turns out to be essential to prove various inequalities, and relates the relative quermassintegral $\mathrm{W}_{i+1}(K_\lambda^p; E)$ and the mixed quermassintegral $\mathrm{W}_{p,i}(K_\lambda^p, E; E)$.

Proposition 4.2. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leq p < \infty$, $\mathrm{r} = \mathrm{r}(K; E)$ and $i \in \{0, 1, \ldots, n-1\}$. Then, for all $\lambda \in (-\mathrm{r}, \infty)$, we have that*

$$(\mathrm{r} +_p \lambda)^{p-1}\mathrm{W}_{p,i}(K_\lambda^p, E; E) \leq \mathrm{W}_{i+1}(K_\lambda^p; E).$$

Proof. Since $(\mathrm{r} +_p \lambda)E \subseteq K_\lambda^p$, and $p \geq 1$, we have that

$$h(K_\lambda^p, u)^{1-p} \leq (\mathrm{r} +_p \lambda)^{1-p} h(E, u)^{1-p}.$$

Thus,

$$\begin{aligned}\mathrm{W}_{p,i}(K_\lambda^p, E; E) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(E, u)^p h(K_\lambda^p, u)^{1-p}\,\mathrm{dS}\big(K_\lambda^p[n-i-1], E[i], u\big) \\ &\leq \frac{1}{n}(\mathrm{r} +_p \lambda)^{1-p}\int_{\mathbb{S}^{n-1}} h(E, u)\,\mathrm{dS}\big(K_\lambda^p[n-i-1], E[i], u\big) \\ &= (\mathrm{r} +_p \lambda)^{1-p}\mathrm{W}_{i+1}(K_\lambda^p; E),\end{aligned}$$

concluding the proof. □

Next we prove the upper bound for the quermassintegrals of the p-inner parallel bodies given in Theorem 1.7.

Proof of Theorem 1.7. The technique of the proof follows the ideas from the one of [3, Theorem 2]. Using Proposition (3.2) iii), the monotonicity and the linearity of the mixed volumes (see e.g. [32, Section 5.1]), and the well-known inclusion

$$K + E \subseteq 2^{(p-1)/p}(K +_p E) \tag{4.2}$$

(see [9, Theorem 1]), we get

$$\begin{aligned}
\mathrm{W}_i(K;E) &= \mathrm{V}\big(K[n-i],E[i]\big) \geq \mathrm{V}\big(K_\lambda^p +_p |\lambda| K_p^*, K[n-i-1], E[i]\big)\\
&\geq \alpha_p \mathrm{V}\big(K_\lambda^p + |\lambda| K_p^*, K[n-i-1], E[i]\big)\\
&= \alpha_p \Big(\mathrm{V}\big(K_\lambda^p, K[n-i-1], E[i]\big) + |\lambda| \mathrm{V}\big(K_p^*, K[n-i-1], E[i]\big)\Big)\\
&\geq \alpha_p |\lambda| \mathrm{V}\big(K_p^*, K[n-i-1], E[i]\big)\\
&\quad + \alpha_p \mathrm{V}\big(K_\lambda^p, K_\lambda^p +_p |\lambda| K_p^*, K[n-i-2], E[i]\big)\\
&\geq \alpha_p |\lambda| \mathrm{V}\big(K_p^*, K[n-i-1], E[i]\big) + \alpha_p^2 \mathrm{V}\big(K_\lambda^p[2], K[n-i-2], E[i]\big)\\
&\quad + |\lambda| \alpha_p \mathrm{V}\big(K_p^*, K_\lambda^p, K[n-i-2], E[i]\big)\\
&\geq \dots\\
&\geq \alpha_p^{n-i} \mathrm{W}_i(K_\lambda^p;E) + \alpha_p |\lambda| \mathrm{V}\big(K_p^*, K[n-i-1], E[i]\big)\\
&\quad + |\lambda| \sum_{j=1}^{n-i-1} \alpha_p^j \mathrm{V}\big(K_\lambda^p[j], K[n-i-j-1], K_p^*, E[i]\big),
\end{aligned}$$

which concludes the proof. □

As a consequence of Theorem 1.7, one can get the following upper bound for the relative quermassintegral $\mathrm{W}_i(K_\lambda^p;E)$ in terms of a finite sum of mixed volumes involving K, E and the p-inner parallel body K_λ^p itself, as we know that $E \subseteq K^* \subseteq K_p^*$ (see Proposition 3.2 ii)).

Theorem 4.3. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \leq p < \infty$, $\mathrm{r} = \mathrm{r}(K;E)$, and let $i \in \{0, 1, \dots, n-1\}$. Then, for all $\lambda \in [-\mathrm{r}, 0]$ we have that*

$$\begin{aligned}
\alpha_p^{n-i} \mathrm{W}_i(K_\lambda^p;E) \leq \mathrm{W}_i(K;E) - |\lambda| \alpha_p \mathrm{W}_{i+1}(K;E) - |\lambda| \alpha_p^{n-i-1} \mathrm{W}_{i+1}(K_\lambda^p;E)\\
- |\lambda| \sum_{j=1}^{n-i-2} \alpha_p^j \mathrm{V}\big(K_\lambda^p[j], K[n-i-j-1], E[i+1]\big),
\end{aligned}$$

where $\alpha_p = 2^{\frac{1}{p}-1}$.

The case $p = 1$ was proved in [3, Theorem 2]. As a by-product of Theorem 4.3, the following result is immediately obtained by considering $i = 0$, which provides us with an upper bound for the volume $\mathrm{vol}(K_\lambda^p) = \mathrm{W}_0(K_\lambda^p;E)$ in terms of $\mathrm{W}_1(K;E)$, $\mathrm{W}_1(K_\lambda^p;E)$ and a finite sum of mixed volumes of K, K_λ^p and the gauge body E. The case $p = 1$ can be found in [3, Corollary 1].

Corollary 4.4. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \leq p < \infty$, and let $\mathrm{r} = \mathrm{r}(K;E)$. Then, for all $\lambda \in [-\mathrm{r}, 0]$ we have that*

$$\begin{aligned}
\alpha_p^n \mathrm{vol}(K_\lambda^p) \leq \mathrm{vol}(K) - |\lambda| \alpha_p \mathrm{W}_1(K;E) - |\lambda| \alpha_p^{n-1} \mathrm{W}_1(K_\lambda^p;E)\\
- |\lambda| \sum_{j=1}^{n-2} \alpha_p^j \mathrm{V}\big(K_\lambda^p[j], K[n-j-1], E\big),
\end{aligned}$$

where $\alpha_p = 2^{\frac{1}{p}-1}$.

Although the results involving the gauge body E are always weaker than the corresponding ones with K^* or K_p^*, one of the advantages of using E is that sometimes the inequalities that appear can be improved using known results for quermassintegrals, that do not exist for arbitrary mixed volumes.

Even though we have not been able to obtain an upper bound for the i-th quermassintegral of K_λ^p, in which no p-inner parallel body appears (cf. Theorems 1.7 and 4.3), in the following result we obtain an upper bound for it in which *only* the relative quermassintegral $\mathrm{W}_i(K;E)$ and a finite sum of mixed volumes of K, $K_{-\mathrm{r}}^p$, K_p^* and E show up. Notice that λ will appear solely as a multiplicative factor in the right-hand side of the inequality in Theorem 4.5, but not in dependence of a convex body.

Theorem 4.5. *Let* $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \le p < \infty$, $\mathrm{r} = \mathrm{r}(K;E)$ *and let* $i \in \{0,1,\dots,n-1\}$. *Then, for all* $\lambda \in (-\mathrm{r},0]$ *we have that*

$$\begin{aligned}\alpha_p^{n-i}\mathrm{W}_i(K_\lambda^p;E) \le{}& \mathrm{W}_i(K;E) - |\lambda|\alpha_p \mathrm{V}(K[n-i-1],K_p^*,E[i])\\ &- |\lambda| \sum_{j=1}^{n-i-1} \alpha_p^{2j} \sum_{k=0}^{j} \binom{j}{k} (\mathrm{r} +_p \lambda)^{j-k} \mathrm{V}_{ijk},\end{aligned} \tag{4.3}$$

where we write $\mathrm{V}_{ijk} := \mathrm{V}\big(K_{-\mathrm{r}}^p[k], K_p^*[j-k+1], K[n-i-j-1], E[i]\big)$ *for the sake of brevity and* $\alpha_p = 2^{\frac{1}{p}-1}$.

Proof. By Remark 3.4 and using the inclusion (4.2), we have that

$$\begin{aligned}&\mathrm{V}\big(K_\lambda^p[j], K[n-i-j-1], K_p^*, E[i]\big)\\ &\quad\ge \alpha_p^j \mathrm{V}\big(K_{-\mathrm{r}}^p + (\mathrm{r} +_p \lambda)K_p^*[j], K[n-i-j-1], K_p^*, E[i]\big)\\ &\quad= \alpha_p^j \sum_{k=0}^{j} \binom{j}{k} (\mathrm{r} +_p \lambda)^{j-k} \mathrm{V}\big(K_{-\mathrm{r}}^p[k], K_p^*[j-k+1], K[n-i-j-1], E[i]\big).\end{aligned}$$

Combining this inequality with Theorem 1.7 we finish the proof. □

We remark again that although the inequality is, because of the use of Remark 3.4, *weaker* than the one in Theorem 1.7, the advantage of the right-hand side of (4.3) relies on the fact that the involved convex bodies do not depend on the parameter λ.

In [15, Theorem 24 and remarks afterwards] the following integral representation for the volume was shown: Let $E \in \mathcal{K}_{(0)}^n$, $K \in \mathcal{K}_{00}^n(E)$, $\mathrm{r} = \mathrm{r}(K;E)$, $1 \le p < \infty$ and let $-\mathrm{r} \le \lambda \le 0$. Then

$$\mathrm{vol}(K) - \mathrm{vol}(K_\lambda^p) = n \int_\lambda^0 |t|^{p-1} \mathrm{W}_{p,0}(K_t^p, E; E)\,\mathrm{d}t. \tag{4.4}$$

We conclude the paper using the above identity in order to provide a further lower bound for the volume of the p-inner parallel bodies of a convex body

K. For the sake of brevity, we write

$$g_p(\lambda) := \int_\lambda^0 \frac{|t|^p}{\left(\mathrm{r}^p - |t|^p\right)^{(p-1)/p}}\,\mathrm{d}t.$$

Theorem 4.6. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $1 \le p < \infty$ and let $\mathrm{r} = \mathrm{r}(K;E)$. Then, for all $\lambda \in (-\mathrm{r}, 0]$ we have that*

$$\mathrm{vol}(K^p_\lambda) \ge \mathrm{vol}(K) + \frac{n}{\alpha_p^{n-2}} g_p(\lambda)\mathrm{W}_2(K;E) - \frac{\mathrm{r} - \left(\mathrm{r}^p - |\lambda|^p\right)^{1/p}}{\alpha_p^{n-1}} n\mathrm{W}_1(K;E),$$

where $\alpha_p = 2^{\frac{1}{p}-1}$.

Proof. Theorem 4.3 with $i = 1$ implies that

$$\alpha_p^{n-1}\mathrm{W}_1(K^p_\lambda;E) \le \mathrm{W}_1(K;E) - |\lambda|\alpha_p \mathrm{W}_2(K;E).$$

This, together with Proposition 4.2 for $i = 0$, yield

$$\begin{aligned} n|\lambda|^{p-1}\mathrm{W}_{p,0}(K^p_\lambda, E; E) \le{}& \frac{n}{\alpha_p^{n-1}} \frac{|\lambda|^{p-1}}{(\mathrm{r} +_p \lambda)^{p-1}} \mathrm{W}_1(K;E) \\ &- \frac{n}{\alpha_p^{n-2}} \frac{|\lambda|^p}{(\mathrm{r} +_p \lambda)^{p-1}} \mathrm{W}_2(K;E). \end{aligned}$$

Then, integrating this expression in $(\lambda, 0)$ and using (4.4) we get

$$\begin{aligned} \mathrm{vol}(K) - \mathrm{vol}(K^p_\lambda) &= n\int_\lambda^0 |t|^{p-1}\mathrm{W}_{p,0}(K^p_t, E; E)\,\mathrm{d}t \\ &\le \frac{n\mathrm{W}_1(K;E)}{\alpha_p^{n-1}} \int_\lambda^0 \frac{|t|^{p-1}}{\left(\mathrm{r}^p - |t|^p\right)^{(p-1)/p}}\,\mathrm{d}t \\ &\quad - \frac{n\mathrm{W}_2(K;E)}{\alpha_p^{n-2}} \int_\lambda^0 \frac{|t|^p}{\left(\mathrm{r}^p - |t|^p\right)^{(p-1)/p}}\,\mathrm{d}t \\ &= \frac{n\mathrm{W}_1(K;E)}{\alpha_p^{n-1}} \left(\mathrm{r} - \left(\mathrm{r}^p - |\lambda|^p\right)^{1/p}\right) - \frac{n\mathrm{W}_2(K;E)}{\alpha_p^{n-2}} g_p(\lambda), \end{aligned}$$

which shows the result. □

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